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Similarity solutions of the entropy transport equation

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ABSTRACT

Individual terms in the entropy transport equation are used today to analyse entropy production and dissipation within the flow. The underlying idea is that the local entropy production tells us where a particular design can be improved. However, very little attention has been paid up to now to the analysis of the full entropy transport equation. The present paper focuses on this topic and shows similarity solutions of the entropy transport equation. Two cases have been investigated as examples: laminar flow over a flat plate and flow between converging and diverging parallel plates, so called Jeffery–Hamel flows. Thus, similarity solutions have been derived for the boundary layer equations as well as for an analytical solution of the Navier–Stokes equations. These solutions show nicely the variation of all different terms in the entropy transport equation and highlight also restrictions under which such solutions can be obtained.

1. Introduction

An efficient use of energy is one of the major objectives in the design of many technical systems today. This can only be achieved, if the Second Law of thermodynamics is taken into account, since the amount of available work is closely linked to the amount of entropy production (see e.g. [1]). Thus, a thermal apparatus producing less entropy by irreversibility destroys less available work. This increases the total efficiency of the thermal system. The amount of entropy produced can be used directly as an efficiency parameter of the system (see e.g. [2]). Second Law and entropy production analysis have therefore been widely used to evaluate the sources of irreversibilities in various components and systems. Some examples are given in [3–6]. Kock and Herwig [4] analysed the entropy production in incompressible turbulent shear flow. They developed wall functions for the entropy production terms and incorporated them into a CFD code. As an example pipe flow with heat transfer was analysed and compared to results from a direct numerical simulation with special emphasis on the entropy production. Andreozzi et al. [5] studied numerically the local and global entropy generation rates in natural convection in air in a vertical channel. Results of entropy generation analysis are obtained by solving the entropy generation equation based on the velocity and temperature data. Abu-Hijleh and Heilen [6] investigated the entropy generation due to laminar natural convection over a heated rotating cylinder. They found that the entropy generation increased as the Reynolds number and buoyancy parameter increased.

The entropy transport equation has long been well-known (see e.g. [7,16]). However, in the past interest mostly focused on some terms in the equation, in particular the entropy production term. To the best knowledge of the present authors no one has investigated similarity solutions of the entropy transport equation. Obviously, there must be some similarity solutions also for this equation, if we assume self-similar flow and self-similar heat transfer. However, it is worth looking at such solutions, because, in deriving similarity solutions, restrictions are normally pointed out for the resulting boundary conditions. In addition, self-similar solutions result in expressions for all terms of the entropy transport equation, solely dependent on one coordinate.

2. Analysis

The thermodynamic quantity entropy is a state variable. For the entropy transport the following differential transport equation can be derived [7,16]

$$\rho\frac{Ds}{Dt} = -\mathrm{div}\left(\frac{q}{T}\right) + \frac{\Phi_{\mathrm{Dis}}}{T} + \frac{\Phi_{\mathrm{Cond}}}{T^2} \tag{1}$$

In Equation (1) the term on the left-hand side of the equation denotes the total change of entropy, the terms on the right-hand side of the equation denote the diffusion, viscous and thermal entropy generation. If we restrict the further investigations to

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Nomenclature

a	thermal diffusivity [m^2/s]
c_p	specific heat at constant pressure [J/(kg K)]
c_v	specific heat at constant volume [J/(kg K)]
C	Chapman–Rubesin parameter, $= \rho\mu/(\rho_\infty\mu_\infty)$ [–]
F, G, H	functions [–]
h_0	specific total enthalpy [J/kg]
K	constant for Jeffery–Hamel flows [–]
k	thermal conductivity [W/(m K)]
Ma_∞	free-stream Mach number, $= u_\infty/\sqrt{\kappa R T_\infty}$ [–]
p	pressure [Pa]
q	heat flux [W/m^2]
Pr	Prandtl number, $= \nu/a$ [–]
r	radial coordinate [m]
R	specific gas constant [J/(kg K)]
s	specific entropy [J/(kg K)]
T	temperature [K]
T_{ref}	reference temperature [K]
u, v, w	velocity components [m/s]
u_∞	free-stream velocity [m/s]
v_r, v_φ	radial and tangential velocity component [m/s]
x, y, z	coordinates [m]

Greek symbols

α	half opening angle of the channel [–]
ρ	density [kg/m^3]
η	similarity coordinate [–]
φ	angle [–]
ϕ	enthalpy ratio [–]
μ	dynamic viscosity [kg/(m s)]
ν	kinematic viscosity, $= \mu/\rho$ [m^2/s]
κ	specific heat ratio, $= c_p/c_v$ [–]
ζ	modified coordinate [kg/(m s)]
Φ_{Dis}	entropy dissipation [W/m^3]
Φ_{Cond}	entropy generation by heat transfer [W K/m^3]
Ψ	stream function [kg/(m s)]

Indices

∞	free-stream
w	at the wall

laminar flows, Equation (1) can be written for a stationary flow in a Cartesian coordinate system (x, y, z) with the velocity components (u, v, w) as follows

$$\rho\left(u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}+w\frac{\partial s}{\partial z}\right)=\frac{\partial}{\partial x}\left(\frac{k\partial T}{T\partial x}\right)+\frac{\partial}{\partial y}\left(\frac{k\partial T}{T\partial y}\right)+\frac{\partial}{\partial z}\left(\frac{k\partial T}{T\partial z}\right)$$
$$+\frac{\mu}{T}\left[2\left(\frac{\partial u}{\partial x}\right)^2+2\left(\frac{\partial v}{\partial y}\right)^2+2\left(\frac{\partial w}{\partial z}\right)^2+\left(\frac{\partial u}{\partial y}+\frac{\partial v}{\partial x}\right)^2\right.$$
$$\left.+\left(\frac{\partial u}{\partial z}+\frac{\partial w}{\partial x}\right)^2+\left(\frac{\partial v}{\partial z}+\frac{\partial w}{\partial y}\right)^2-\frac{2}{3}\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}+\frac{\partial w}{\partial z}\right)^2\right]$$
$$+\frac{k}{T^2}\left[\left(\frac{\partial T}{\partial x}\right)^2+\left(\frac{\partial T}{\partial y}\right)^2+\left(\frac{\partial T}{\partial z}\right)^2\right] \tag{2}$$

For a two-dimensional boundary layer type flow, with the main flow, u, in x-direction, Equation (2) simplifies by invoking the usual boundary layer simplifications [8] to

$$\rho\left(u\frac{\partial s}{\partial x}+v\frac{\partial s}{\partial y}\right) = \frac{\partial}{\partial y}\left(\frac{k}{T}\frac{\partial T}{\partial y}\right)+\frac{\mu}{T}\left(\frac{\partial u}{\partial y}\right)^2+\frac{k}{T^2}\left(\frac{\partial T}{\partial y}\right)^2 \tag{3}$$

Because the entropy is a thermodynamic state variable, we always have for a simple system in addition to Equation (3) a relation between the entropy and two other state variables, like

$$s = s(p, T) \tag{4}$$

For an ideal gas, this might result in the equation

$$\mathrm{d}s = c_v\frac{\mathrm{d}T}{T}-R\frac{\mathrm{d}\rho}{\rho}, \tag{5}$$

where for an incompressible flow, with $\mathrm{d}\rho = 0$, the second term on the right-hand side of equation (5) is zero and also $c_v = c_p = c$. By using equations (4) and (5), the flow is assumed to evolve through a series of quasi-equilibrium states.

In the following, consideration is given to flow over a flat plate, which is a boundary layer type flow and to a flow through a converging or diverging planar nozzle (Jeffery–Hamel flow), where the flow permits an analytical solution of the full Navier–Stokes equations.

2.1. Flow over a flat plate

The flow over a flat plate is sketched in Fig. 1. In the following, consideration is given to the case of a compressible flow over a flat plate. For this case similarity solutions for the entropy transport equation will be derived. Let us consider a laminar, compressible flow over a flat plate. The boundary layer equations are given by (see e.g. Kays et al. [8] or Weigand [9])

$$\frac{\partial}{\partial x}(\rho u)+\frac{\partial}{\partial y}(\rho v) = 0 \tag{6}$$

$$\rho u\frac{\partial u}{\partial x}+\rho v\frac{\partial u}{\partial y} = -\frac{\mathrm{d}p}{\mathrm{d}x}+\frac{\partial}{\partial y}\left(\mu\frac{\partial u}{\partial y}\right) \tag{7}$$

$$\rho\left(u\frac{\partial h_0}{\partial x}+v\frac{\partial h_0}{\partial y}\right) = \frac{\partial}{\partial y}\left(\mu\frac{Pr-1}{Pr}u\frac{\partial u}{\partial y}\right)+\frac{\partial}{\partial y}\left(\frac{\mu}{Pr}\frac{\partial h_0}{\partial y}\right) \tag{8}$$

where h_0 is the specific total enthalpy, which is defined for a boundary layer flow ($v \ll u$) by

$$h_0 = c_pT+\frac{1}{2}\left(u^2+v^2\right) = c_pT+\frac{1}{2}u^2 \tag{9}$$

The system of equations (6)–(8) is strongly coupled by density. In addition, the fluid properties may no longer be considered constant, due to the high velocities under consideration. It can be assumed that the dynamic viscosity can be approximated by

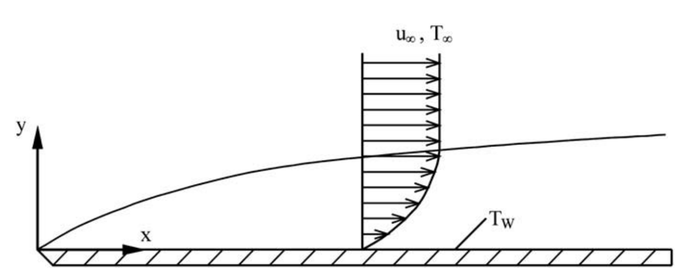

Fig. 1. Geometrical configuration and coordinate system for a flow over a flat plate.

$$\frac{\mu(T)}{\mu\left(T_{\text{ref}}\right)} = \left(\frac{T}{T_{\text{ref}}}\right)^{\omega}, \quad 0.5 \le \omega \le 1 \tag{10}$$

where T_{ref} is a reference temperature. The density is related to the pressure and the temperature by a thermal state equation and we assume that the fluid under consideration can be considered as an ideal gas. Thus,

$$\rho = \frac{p}{RT} \tag{11}$$

In addition, we assume that c_p is constant. The set of partial differential equations (6)–(8) has to be solved together with equations (10) and (11) and the following boundary conditions

$$\begin{aligned} &x = 0 : u, v, h_0 \quad \text{given} \\ &y = 0 : u = v = 0, \quad h_0 = h_W \\ &y \to \infty : u = u_\infty(x), \quad h_0 = h_{0\infty} \end{aligned} \tag{12}$$

It is convenient to introduce a stream function into the compressible boundary layer equations defined by

$$\rho u = \frac{\partial \Psi}{\partial y}, \quad \rho v = -\frac{\partial \Psi}{\partial x} \tag{13}$$

In addition, the following new coordinates are commonly introduced

$$\zeta = \rho_\infty u_\infty \int_0^y \frac{\rho}{\rho_\infty} \, d\bar{y}, \quad \bar{x} = \int_0^x \rho_\infty u_\infty \eta_\infty \, d\bar{x} \tag{14}$$

Similarity solutions of the compressible boundary layer equations have been investigated for example by Li and Nagamatsu [11] and a good overview on this subject can be found for example in Schlichting [10]. One can derive the similarity equations for the flow and the temperature field by introducing the following new variables into equations (6)–(8)

$$\eta = \frac{\varsigma}{\sqrt{2\bar{x}}}, \quad \Psi = \sqrt{2\bar{x}} f(\eta), \quad \phi = \frac{h_0}{h_{0\infty}} \tag{15}$$

This results in

$$(Cf'')' + \frac{2\bar{x}}{u_\infty}\frac{du_\infty}{d\bar{x}}\left(\frac{\rho_\infty}{\rho} - (f')^2\right) + ff'' = 0 \tag{16}$$

$$\left(\frac{C}{Pr}\phi'\right)' + f\phi' + \frac{u_\infty}{h_{0\infty}}\left(\frac{Pr - 1}{Pr} Cf'f''\right) = 0 \tag{17}$$

These two coupled ordinary differential equations have to be solved together with the following boundary conditions

$$\begin{aligned} &\eta = 0 : \quad f(0) = 0, \; f'(0) = 0, \; \phi(0) = \phi_W \\ &\eta \to \infty : \quad f'(\infty) = 1, \; \phi(\infty) = 1 \end{aligned} \tag{18}$$

The quantity C, which appears in the above equations, is the Chapman–Rubesin parameter, which can be described as a function of temperature by using equations (10) and (11).

$$C = \frac{\rho\mu}{\rho_\infty \mu_\infty}, \quad Pr = \frac{\mu c_p}{k},$$

$$C = \frac{p}{RT}\frac{RT_\infty}{p_\infty}\left(\frac{T}{T_\infty}\right)^{\omega} = \left(\frac{T}{T_\infty}\right)^{\omega - 1}, \quad 0.5 \le \omega \le 1 \tag{19}$$

In order to guarantee that the equations (16) and (17) form a set of ordinary differential equations, any dependence on $\bar{x}$ must cancel out from these equations. This fact leads to the following restrictions on the coefficients in these equations:

- The Prandtl number must be a constant or only depend on η.
- The expression $(2\bar{x}/u_\infty)(du_\infty/d\bar{x}) = \beta$ has to be constant, which means that $u_\infty = C_1\bar{x}^m$.
- Finally, the expression $(u_\infty^2/h_{0\infty}) = 2(1 + (2/(\kappa - 1)Ma_\infty^2))^{-1}$ has to be constant, or the Prandtl number must be equal to one. For most gases the assumption that $Pr = 1$ is reasonable. If this is not the case, then either the Mach number ($Ma_\infty = u_\infty/\sqrt{\kappa R T_\infty}$) is constant or the Mach number is sufficiently large, so that $u_\infty^2/h_{0\infty}$ can be approximated by 2.

Let us now introduce the similarity variables, with which one can obtain similarity solutions for the flow and enthalpy field, into the entropy transport equation (3). For the individual terms in equation (3) one obtains after some algebra:

Convection term:

$$\rho\left(u\frac{\partial s}{\partial x} + v\frac{\partial s}{\partial y}\right) = -c_p D f \rho_\infty^2 u_\infty^2 \mu_\infty \frac{1}{2\bar{x}}\frac{\partial \tilde{s}}{\partial \eta} \tag{20}$$

Diffusion term:

$$\begin{aligned} \frac{\partial}{\partial y}\left(\frac{k}{T}\frac{\partial T}{\partial y}\right) = {} & \frac{c_p^{1-\omega}}{Pr} D E^{-\omega} \rho_\infty^2 u_\infty^{2\omega+2} T_\infty^{-\omega} \mu_\infty \frac{1}{2\bar{x}} \left(\phi - \frac{1}{2}(f')^2 E\right)^{\omega - 1} \\ & \left\{\frac{2(\phi' - f'f''E)^2}{2\phi - (f')^2 E}(\omega - 1)D + (\phi' D)' - E[Df'f'']'\right\} \end{aligned}$$

Dissipation term:

$$\frac{\mu}{T}\left(\frac{\partial u}{\partial y}\right)^2 = c_p CDE (f'')^2 \rho_\infty^2 u_\infty^2 \mu_\infty \frac{1}{\bar{x}} \frac{1}{2\phi - (f')^2 E} \tag{22}$$

Production term by heat transfer:

$$\frac{k}{T^2}\left(\frac{\partial T}{\partial y}\right)^2 = \frac{c_p}{Pr} CD \rho_\infty^2 u_\infty^2 \mu_\infty \frac{1}{2\bar{x}} \left(\frac{2\phi' - 2f'f''E}{2\phi - (f')^2 E}\right)^2 \tag{23}$$

where the following abbreviations have been used

$$D = \frac{\rho}{\rho_\infty}, \; E = \frac{u_\infty^2}{h_{0\infty}}, \; f'(\eta) = \frac{u}{u_\infty}, \; C = \frac{\rho\mu}{\rho_\infty \mu_\infty}, \; \tilde{s} = \frac{s - s_\infty}{c_p} \tag{24}$$

Combining the equations (20)–(23) and inserting into equation (3) results in

$$\begin{aligned} -Pr\, f \frac{\partial \tilde{s}}{\partial \eta} = {} & c_p^{-\omega} E^{-\omega} u_\infty^{2\omega} T_\infty^{-\omega} \left(\phi - \frac{1}{2}(f')^2 E\right)^{\omega - 1} \\ & \left\{\frac{2(\phi' - f'f''E)^2}{2\phi - (f')^2 E}(\omega - 1)D + (\phi' D)' - E[Df'f'']'\right\} \\ & + 2Pr\, CE(f'')^2 \frac{1}{2\phi - (f')^2 E} + C\left(\frac{2\phi' - 2f'f''E}{2\phi - (f')^2 E}\right)^2 \end{aligned} \tag{25}$$

The entropy transport equation (25) has to be solved with the boundary condition

$$\tilde{s} = 0, \quad \text{for } \eta \to \infty \tag{26}$$

In the above equations the prime denotes the derivative of the quantity with respect to the similarity variable η. In order to guarantee that equation (25) is a similarity equation, no dependence on $\bar{x}$ is allowed. This means that E must be constant. Having $h_{0\infty} = \text{const.}$, this is only possible if $u_\infty = \text{const.}$ This means that one gets a stronger restriction for similarity solutions for the entropy transport equation than for the energy equation (where we had the choice to assume that either $E = \text{const.}$ (and therefore $u_\infty = \text{const.}$) or that $\Pr = 1$.

Similarity solutions for the entropy transport equation for an incompressible flow over a flat plate can be obtained analogously.

2.2. Jeffery–Hamel flows

We next develop a similarity solution of the entropy transport equation in the full Navier–Stokes equations, describing the flows between convergent and divergent flat plates as shown in Fig. 2. Detailed discussions of these Jeffery–Hamel flows can be found in [10,12–14]. Here, the velocity and the temperature distributions depend only on the angle φ, i.e. the tangential velocity component $v_\varphi = 0$ and the radial velocity component v_{r} is given by [10]

$$v_{\mathrm{r}} = \frac{\mu}{\rho r} F(\varphi) \tag{27}$$

Inserting the results for v_φ and v_{r} into the Navier–Stokes and the energy equations result in ordinary differential equations for the velocity field ($F(\varphi)$) and temperature field ($G(\varphi)$). The function $F(\varphi)$ can be obtained from the solution of the following ordinary differential equation

$$F^2 + 4F + F'' + K = 0, \quad F(\pm\alpha = 0) \tag{28}$$

which has been obtained from the momentum equations for an incompressible fluid [10]. The quantity K is a constant and is given by the radial pressure gradient ($K = -r^3/(\rho\nu^2)\, \mathrm{d}p_{\mathrm{W}}/\mathrm{d}r$).

The temperature field is also only dependent on the angle φ and has the form [10]

$$\frac{T - T_{\mathrm{W}}}{\mu^2/\rho^2} c_v = \frac{1}{r^2} G(\varphi) \tag{29}$$

Here the function $G(\varphi)$ is the solution of the following ordinary differential equation

$$G'' + G(4 + 2Pr\, F) + Pr\left(4F^2 + (F')^2\right) = 0, \quad G(\pm\alpha = 0) \tag{30}$$

which has been obtained from the energy equation [10]. The entropy transport equation (2) has to be transformed into polar coordinates, because the velocity and temperature field for these types of flows are already known in these coordinates. One obtains for Jeffery–Hamel flows after some effort

$$\rho v_{\mathrm{r}} \frac{\partial s}{\partial r} = \frac{k}{T}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial T}{\partial \varphi}\right) - \frac{k}{T^2}\left(\left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \varphi}\right)^2\right) + \frac{\mu}{T}\left(2\left(\frac{\partial v_{\mathrm{r}}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial v_{\mathrm{r}}}{\partial \varphi}\right)^2 + 2\frac{v_{\mathrm{r}}^2}{r^2}\right) + \frac{k}{T^2}\left[\left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \varphi}\right)^2\right] \tag{31}$$

In this equation, the term on the left-hand side of equation (31) represents the entropy convection, while the first two terms on the right-hand side are the diffusion terms. The third term is the entropy production by viscous dissipation and the last term is the production term by heat transfer.

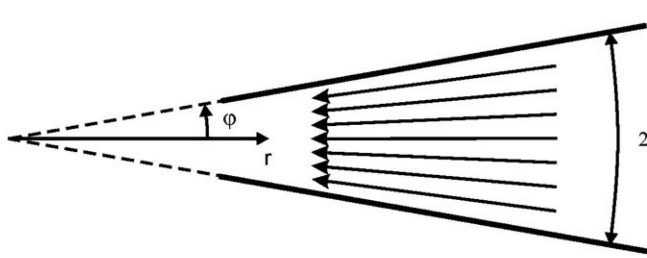

Fig. 2. Geometrical configuration and coordinate system for Jeffery–Hamel flows.

Let us now introduce the similarity variables, with which one can obtain similarity solutions for the flow field and temperature field, into the entropy transport equation (31). For the individual terms in equation (31) one obtains, after some algebra:

Convection term:

$$\rho v_{\mathrm{r}} \frac{\partial s}{\partial r} = \frac{\mu}{r} F \frac{\partial s}{\partial r} \tag{32}$$

Diffusion term:

$$\frac{k}{T}\left(\frac{\partial^2 T}{\partial r^2} + \frac{1}{r}\frac{\partial T}{\partial r} + \frac{1}{r^2}\frac{\partial T}{\partial \varphi}\right) - \frac{k}{T^2}\left(\left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \varphi}\right)^2\right) = \frac{k}{T}\left(\frac{\nu^2}{c_v}\frac{1}{r^4}(4G + G'')\right) - \frac{k}{T^2}\left(\frac{\nu^4}{c_v^2}\frac{1}{r^6}\left(4G^2 + (G')^2\right)\right) \tag{33}$$

Dissipation term:

$$\frac{\mu}{T}\left(2\left(\frac{\partial v_{\mathrm{r}}}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial v_{\mathrm{r}}}{\partial \varphi}\right)^2 + 2\frac{v_{\mathrm{r}}^2}{r^2}\right) = \frac{\mu}{T}\left(\frac{\nu^2}{r^4}\left(4F^2 + (F')^2\right)\right) \tag{34}$$

Production term by heat transfer:

$$\frac{k}{T^2}\left[\left(\frac{\partial T}{\partial r}\right)^2 + \frac{1}{r^2}\left(\frac{\partial T}{\partial \varphi}\right)^2\right] = \frac{k}{T^2}\frac{\nu^4}{c_v^2}\frac{1}{r^6}\left(4G^2 + (G')^2\right) \tag{35}$$

Introducing the individual terms into the entropy transport equation (31) results in

$$\rho T F \frac{\partial s}{\partial r} = \frac{k\nu}{c_v}\frac{1}{r^3}(4G + G'') + \frac{\mu\nu}{r^3}\left(4F^2 + (F')^2\right) \tag{36}$$

For equation (36) similarity solutions for the entropy field can only be found if both sides of the equation are only dependent on φ and not on r. Let us assume a functional relationship for the entropy, given by

$$s - s_0 = c_v H(\varphi) \ln\frac{T}{T_0} \tag{37}$$

One can show that this satisfies equation (36) and, after some algebra, one obtains $H(\varphi) = 1$, i.e. the solution of equation (36) for similarity solutions is equal to

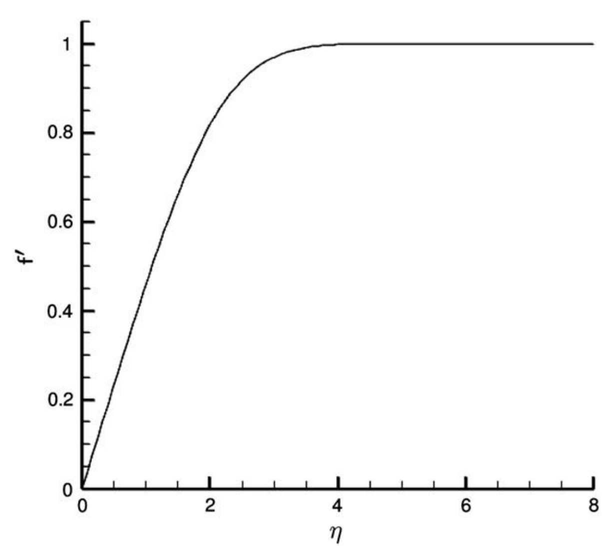

Fig. 3. Velocity distribution u/u_∞ as a function of η for a flow over a flat plate.

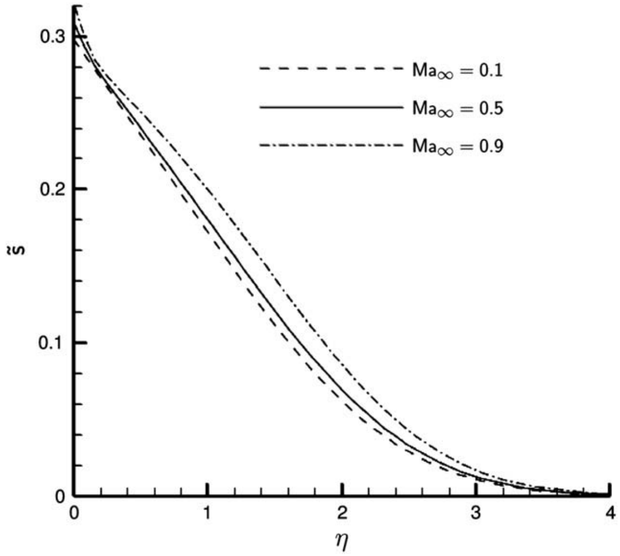

Fig. 4. Distribution of the dimensionless entropy difference as a function of η for a flow over a flat plate.

$$s - s_0 = c_v \ln\frac{T}{T_0} \tag{38}$$

However, this is the state equation for the entropy for an incompressible flow ($c_v = c_p = c$). This shows that the entropy difference can be predicted locally by this equation. However, equation (38) cannot tell us the individual entropy distributions of diffusion, production and dissipation.

3. Results and discussion

The real advantage of the similarity solutions of the entropy transport equations is that all individual terms in equation (1) like convection, diffusion, dissipation and production by heat transfer are only functions of the similarity coordinate. This means that we can visualize them very easily across the boundary layer and in the whole flow domain. Furthermore, these terms can be used to analyse the loss mechanisms in these types of flows. In the following distributions of these individual terms are plotted as functions of the similarity coordinate for the two investigated flows.

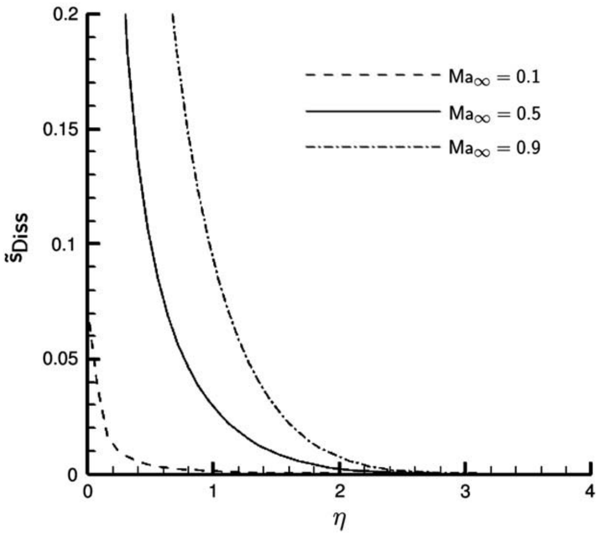

Fig. 6. Distribution of the dissipation term of the entropy transport equation as a function of η for a flow over a flat plate.

3.1. Flow over a flat plate

The similarity equations for the flow and temperature, which are given by equations (16) and (17), can easily be solved numerically by a shooting method, using a fourth order Runge–Kutta method [15]. The distribution of the entropy and all individual terms can then be obtained from equations (20)–(25). The following results have been obtained as an example for the quantities $T_W = 388.15$ K, $T_\infty = 288.15$ K, $\rho_\infty = 1.225\ \text{kg/m}^3$, $\mu_\infty = 17.1 \times 10^{-6}\ \text{kg/(m s)}$, $Pr = 1$, $c_p = 1004.5\ \text{J/(kg K)}$ and various values for u_∞, which are considered to be typical data for a gas flow. Fig. 3 shows the velocity distribution within the boundary layer ($f' = u/u_\infty$) as a function of the similarity coordinate η. It can be

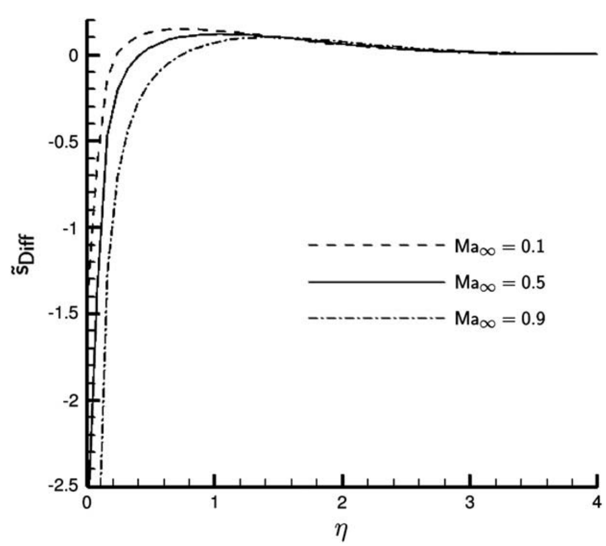

Fig. 5. Distribution of the diffusion term of the entropy transport equation as a function of η for a flow over a flat plate.

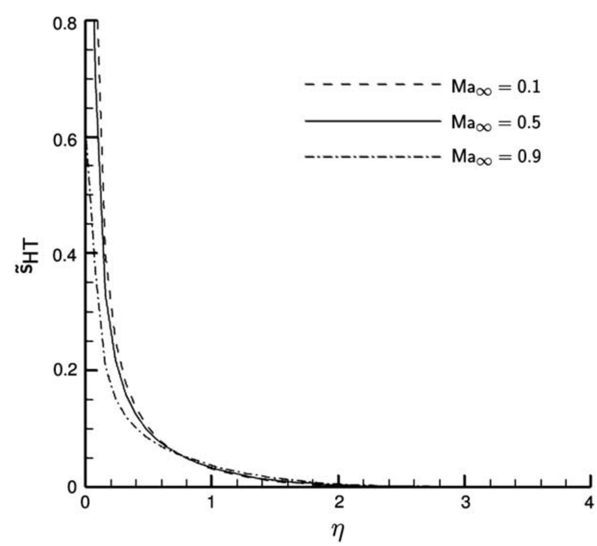

Fig. 7. Distribution of the production term by heat transfer of the entropy transport equation as a function of η for a flow over a flat plate.

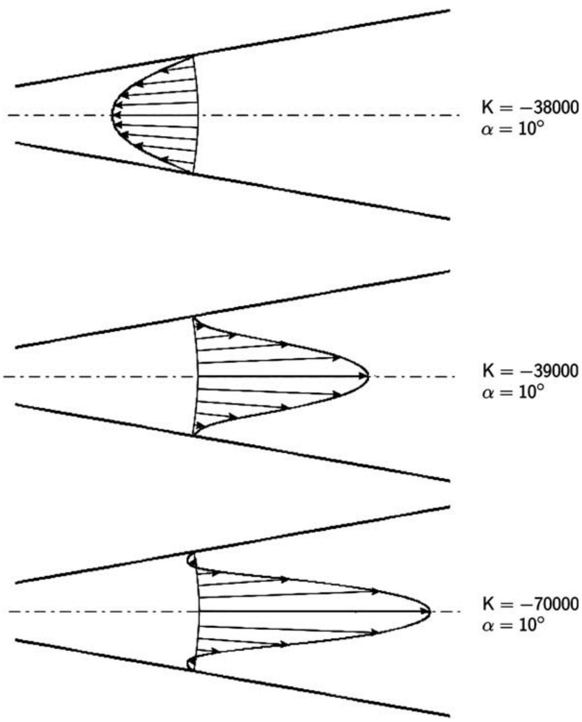

Fig. 8. Velocity fields between convergent and divergent parallel plates for $\alpha = 10^\circ$ and $K = -38{,}000$, $K = -39{,}000$ and $K = -70{,}000$.

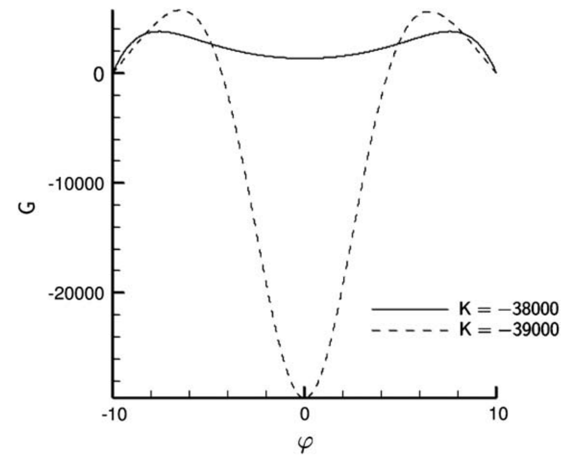

Fig. 10. Dimensionless temperature distribution $G(\varphi)$ for $\alpha = 10^\circ$ and different values of K.

seen that the flow velocity approaches the velocity of the free-stream for large values of η.

Fig. 4 shows the distribution of the dimensionless entropy difference across the boundary layer. The dimensionless entropy difference has a monotonically decreasing shape and achieves the boundary condition, given by equation (26) for large values of the similarity coordinate. It is obvious from this figure that the dimensionless entropy difference only weakly depends on Ma_∞, the Mach number of the flow. This result is due to the used similarity coordinates given by equations (14) and (15). This coordinate transformation is known to transform the boundary layer equations for compressible flow into a form similar to those for an incompressible flow. This means that compressibility effects denoted by the Mach number are already taken into account by the changed coordinates and are, therefore, not strongly visible.

Figs. 5–7 show the individual terms in the entropy transport equation (diffusion, dissipation and production by heat transfer). It is interesting to note that the dissipation and heat conduction terms have simple decreasing shapes, whereas, in the diffusion term a local maximum takes place at about $\eta = 1$, except for $Ma_\infty = 0.9$. In general, the entropy diffusion shows large gradients near the wall, which are clearly driven by the steep velocity and temperature gradients near the wall. The difference between the contours due to Mach number is small. This again might be explained by the similarity coordinates used. For the dissipation term, shown in Fig. 6, one sees that increasing Mach numbers seem to move the region of high entropy dissipation away from the wall. For $Ma_\infty = 0.9$, for example, large values of $\tilde{s}_{\mathrm{Diss}}$ appear for $\eta \approx 1$, whereas for $Ma_\infty = 0.1$ large values of $\tilde{s}_{\mathrm{Diss}}$ appear closer towards the wall. For the entropy production by heat transfer (Fig. 7), no significant change in the contours with changing Mach numbers can be observed. This shows that this term is driven by the large temperature gradients near the wall. Changes with growing Mach numbers are again taken into account by using the transformed coordinates.

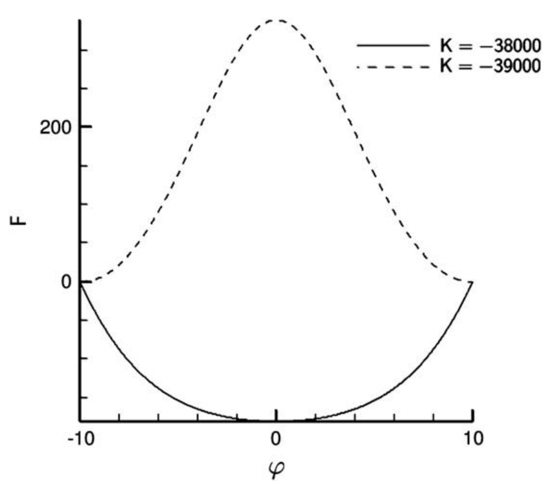

Fig. 9. Dimensionless velocity distribution $F(\varphi)$ for $\alpha = 10^\circ$ and different values of K.

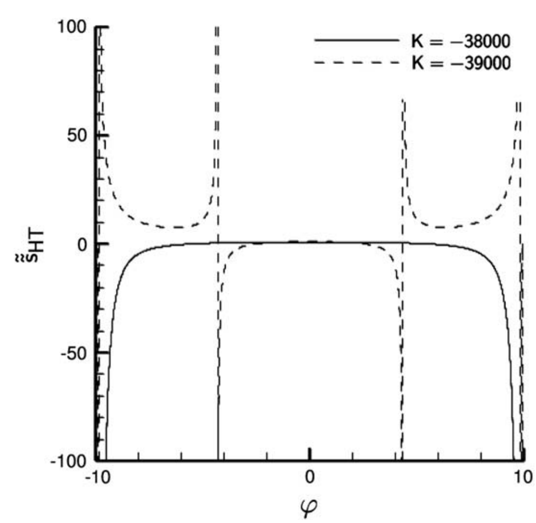

Fig. 11. Distribution of the production term by heat transfer of the entropy transport equation for Jeffery–Hamel flows for $\alpha = 10^\circ$ and different values of K.

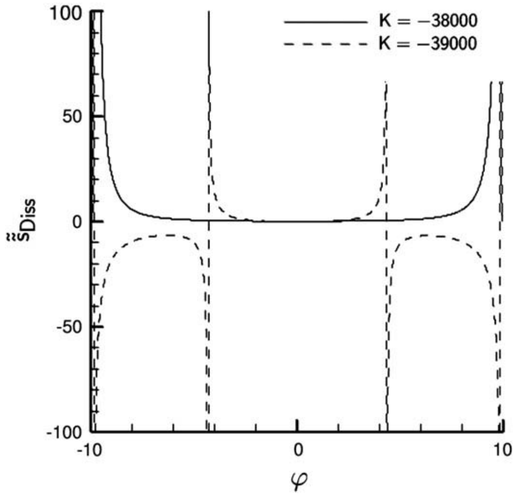

Fig. 12. Distribution of the dissipation term of the entropy transport equation for Jeffery–Hamel flows for $\alpha = 10^{\circ}$ and different values of K.

3.2. *Jeffery–Hamel flows*

Jeffery–Hamel flows are interesting, because different flow structures can be obtained by changing the convergence angle α of the planar channel and the constant K in equation (28). Fig. 8 shows three different examples for the flow fields between converging and diverging planar channels. For the flow parameters set in Fig. 8a one can see that a nozzle flow is obtained. Fig. 8b shows a flow in a diverging channel where the velocity profile has a zero tangent at the wall, whereas in Fig. 8c the flow in the diffusor is already separated near the wall.

In Fig. 9 the velocity distributions are plotted as $F(\varphi) = rv_r/\nu$ for a channel with an opening angle of $\alpha = 10^{\circ}$ as a function of the similarity coordinate φ. One sees from Fig. 9 that the nozzle flow, depicted in Fig. 8a, results in a form for the function F, which is without deflection points. Furthermore, the function F attains negative values for these kinds of flows, which indicates the flow "into the point source at $r \to 0$". For $K = -39{,}000$ we obtain the flow field shown schematically in Fig. 8b. In Fig. 9 one sees very clearly the zero tangent of the velocity near the walls for $\alpha = \pm 10^{\circ}$. Fig. 10 shows the temperature distributions $G(\varphi) = c_v r^2/\nu^2 (T - T_W)$ for the flow fields depicted in Fig. 9. It is very interesting to note that the temperature profile in the diffuser $G(\varphi)$ for $K = -39{,}000$ changes sign for $\varphi \approx \pm 4^{\circ}$, whereas this cannot be observed for the non-dimensional temperature profile in the nozzle flow ($K = -38{,}000$). This change in sign of the local temperature profile means also that the direction of heat flux is inverted between $0 \le \varphi \le 4.1$ and $\varphi \ge 4.1^{\circ}$. The individual terms of the entropy transport equation are displayed in Figs. 11 and 12. Fig. 11 shows the production term for heat transfer for different constants K. It can be seen that for $K = -39{,}000$ singularities appear in the distribution of this term. This is caused for positions in the temperature field, where the temperature differences are zero. Analogously, Fig. 12 shows the distribution of the dissipation term. It can be noticed that the dissipation term is maximum near the walls. Both the production term by heat transfer and the dissipation term have singularities at the positions where the dimensionless temperature is zero. This shows the interesting result that both these terms attain now large values near the wall, but also near positions where the direction of heat flux changes. This indicates that such positions might always be related to high losses and should be avoided. In addition, it seems so, that the appearance of these singularities might be associated with the fact that the flow shown here is unstable and nearly at the limit of flow separation.

4. Conclusions

The present paper addresses similarity solutions of the entropy transport equation for laminar flows. Solutions have been shown for two different flow cases, the flow over a flat plate and for flows between convergent and divergent parallel plates. From the results obtained, we draw the following conclusions:

- Similarity solutions of the entropy transport equations can be obtained for similar flow and temperature fields. However, the entropy transport equations might permit further restrictions in obtaining also a similar entropy field.
- The similarity solutions of the entropy transport equation are very helpful in examining the individual terms in the entropy transport equation and help therefore to better understand their behaviour.

The shown procedure can easily been extended for other similar flow and temperature fields. The here obtained similarity solutions might be very useful for analysing and validation of experimental results on entropy production in boundary layers or in ducts. Furthermore they can be used to check CFD codes for calculating local entropy production.

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